Math 524 Exam 9 Solutions

The first two problems concern Euclidean $\mathbb{R}^3$. Let W be the subspace spanned by $u = (1,0,1)^T$ and $v = (1,2,3)^T$.

1. Find a basis for $W^\perp$.

   It's easiest to use the method from exercise 4 in section 6.6. First, extend $\{u,v\}$ to $\{u,v,w\}$, a basis for $\mathbb{R}^3$. Then, apply Gram-Schmidt to get orthogonal basis $\{u',v',w'\}$. $\{u',v'\}$ will be an orthogonal basis for W, and $\{w'\}$ a (orthogonal) basis for $W^\perp$. Any w will do, so long as the set $\{u,v,w\}$ is independent. I choose $w = (0,1,0)^T$; $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ so these three vectors are independent. $u' = u = (1,0,1)^T$, $v' = v - \frac{\langle u'|v\rangle}{\langle u'|u'\rangle}u' = v - \frac{u'\cdot v}{u'\cdot u'}u' = v - \frac{4}{2}u = (1,2,3)^T - 2(1,0,1)^T = (-1,2,1)^T$, $w' = w - \frac{u'\cdot w}{u'\cdot u'}u' - \frac{v'\cdot w}{v'\cdot v'}v' = w - \frac{0}{2}u' - \frac{2}{6}v' = (0,1,0)^T - (-1/3, 2/3, 1/3)^T = (1/3, 1/3, -1/3)^T$. Hence a basis for $W^\perp$ is $\{(1/3, 1/3, -1/3^T)\}$ or (clearing fractions) $\{(1,1,-1)^T\}$. To check, one can verify that $u'\cdot w' = v'\cdot w' = 0$; $W^\perp$ is one-dimensional because $\mathbb{R}^3 = W \oplus W^\perp$.

2. Write $x = (1,1,1)^T$ as the sum of an element of W and an element of $W^\perp$.

   We project x onto W; however it is important to use an orthogonal basis, such as $\{u',v'\}$ calculated previously, and not the original basis $\{u,v\}$. $Pr_W x = \frac{u'\cdot x}{u'\cdot u'}u' + \frac{v'\cdot x}{v'\cdot v'}v' = \frac{2}{2}(1,0,1)^T + \frac{2}{6}(-1,2,1)^T = (2/3, 2/3, 4/3)^T$. We calculate $x - Pr_W x = (1,1,1)^T - (2/3, 2/3, 4/3)^T = (1/3, 1/3, -1/3)^T$. Hence $x = (2/3, 2/3, 4/3)^T + (1/3, 1/3, -1/3)^T$, as desired.

3. Even quadratic polynomials are of the form $p(x) = \alpha x^2 + \beta$. Find the even quadratic polynomial that best fits (in the sense of least squares) the data $(0,4), (1,-1), (2,10)$.

   We seek a least-squares solution to $Ax = b$, for $A = \begin{pmatrix} 0^2 & 1 \\ 1^2 & 1 \\ 2^2 & 1 \end{pmatrix}$, $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ -1 \\ 10 \end{pmatrix}$. By Thm. 6.7 in the text, this is equivalent to finding a solution to $A^TAx = A^Tb$. We have $A^TA = \begin{pmatrix} 17 & 5 \\ 5 & 3 \end{pmatrix}$, $A^Tb = \begin{pmatrix} 39 \\ 13 \end{pmatrix}$. $\begin{pmatrix} 17 & 5 & 39 \\ 5 & 3 & 13 \end{pmatrix} \to \begin{pmatrix} 7 & -1 & 13 \\ 5 & 3 & 13 \end{pmatrix} \to \begin{pmatrix} 7 & -1 & 13 \\ 26 & 0 & 52 \end{pmatrix}$. Hence $\alpha = 2$ and $\beta = 1$, and the best-fit even quadratic polynomial is $p(x) = 2x^2 + 1$.

4. Carefully state the definition of $\ell_2(\mathbb{R})$. Give two sample vectors.

   $\ell_2(\mathbb{R})$ is the vector space that consists of all infinite sequences of real numbers $(x_1, x_2, \ldots)$ such that the infinite sum $|x_1|^2 + |x_2|^2 + \cdots$ converges. Many sample vectors are possible, of course, such as the standard basis $e_i = (0,0,\ldots,0,1,0,\ldots)$ (1 is in the i^{th} coordinate), or $x = (1, 1/2, 1/3, \ldots)$.